B.Sc. (Math) part III
paper - VI
Topic Conjugacy (Group theory)

Conjugate elements

Def:- Let a, b be two elements of a group G. Then b is said to be conjugate to a if there exists an element $x \in G$ such that
$b = x^{-1}ax$

If $b = x^{-1}ax$ then b is also called the transform of a by x

If b is conjugate to a we write symbolically as $b \sim a$
Thus $b \sim a$ iff $b = x^{-1}ax$ for some $x \in G$
This relation in G is called the relation of conjugacy.

Theorem: The relation of conjugacy is an equivalence relation

Proof:- we have to prove

that the relation conjugacy is (i) Reflexive (ii) Symmetric (iii) Transitive

(i) Reflexive

let a be any element of G then there exists e in G such that

$a = e^{-1}ae$ and hence by definition $a \sim a$

(ii) Symmetric

let $a \sim b$ and we shall show $b \sim a$

since $a \sim b \Rightarrow a = x^{-1}bx$ for some x in G

$\Rightarrow xax^{-1} = x(x^{-1}bx)x^{-1}$

$\Rightarrow xax^{-1} = (xx^{-1})b(xx^{-1}) = b$

$\Rightarrow b = (x^{-1})^{-1}ax^{-1}$ where $x^{-1} \in G \Rightarrow b \sim a$

(iii) Transitivity

let $a \sim b, b \sim c$ we need to prove that $a \sim c$

since $a \sim b$ we have $a = x^{-1}bx$

and since $b \sim c$ we have

$b = y^{-1}cy$ for some $x, y \in G$

Hence we get

$a = x^{-1}(y^{-1}cy)x \therefore b = y^{-1}cy$

$= \cancel{x^{-1}(y^{-1}cy)}$

$= (x^{-1}y^{-1})c(yx)$

$= (yx)^{-1}c(yx) \because$ in a group $(yx)^{-1} = x^{-1}y^{-1}$

$\Rightarrow a = z^{-1}cz$ where $z = (yx) \in G$

Therefore $a \sim c$ and hence the relation $\sim$ is transitive

Hence the relation ~~is~~ of conjugacy in a group G is an equivalence relation.

Conjugate class of an element a

This class will consist of those elements of G which conjugate to a only will be denoted by $C(a)$.

$\therefore C(a) = \{x \in G \mid x \sim a\}$

$= \{x \in G \mid x = y^{-1}ay, y \in G\}$

$= \{y^{-1}ay \mid y \in G\}$

= set of all conjugates of a in G.

If G is a finite group then the number of distinct elements in C(a) will be denoted by C_a